JOURNAL OF APPROXIMATION THEORY 40, 226–241 (1984)

On the Rate of Convergence of the Szász–Mirakyan Operator for Functions of Bounded Variation

FUHUA CHENG

Department of Mathematics, Ohio State University, Columbus, Ohio 43210, U.S.A.*

Communicated by R. Bojanic

Received August 16, 1982

1. INTRODUCTION

Let f be a function defined on the infinite interval $[0, \infty)$. The Szász–Mirakyan operator S_n applied to f is

$$S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_k(nx), \tag{1.1}$$

where

$$p_k(t) = e^{-t} \frac{t^k}{k!}.$$

In 1950 Szász [1] proved[1]:

THEOREM A. *If $f(t)$ is bounded in every finite subinterval of $[0, \infty)$, is equal to $O(t^k)$ for some $k > 0$ as $t \to \infty$, and is continuous at the point $t = x$, then $S_n(f, x)$ converges uniformly to $f(x)$.*

Later on, in 1971, Gróf [3] gave the following estimate for the rate of convergence of $S_n(f, x)$ *when $f(t)$ is continuous on* $[0, \infty)$:

[1] Mirakyan [2] considered the partial sum $\psi_{m,n}(f, x)$ of $S_n(f, x)$,

$$\psi_{m,n}(f, x) = \sum_{k=0}^{m} f\left(\frac{k}{n}\right) P_k(nx)$$

and proved that $\lim_{n\to\infty} \psi_{m,n}(f, x) = f(x)$ uniformly in $[0, r']$, if $\lim_{n\to\infty} (m/n) = r > r' > 0$. We do not consider this matter in this paper.

* Present address: Institute of Computer and Decision Sciences, National Tsing Hua University, Hsinchu, Taiwan 300, R.O.C.

0021-9045/84 $3.00

THEOREM B. *If f is continuous on $[0, \infty)$ and is equal to $O(e^{\alpha x})$ for some $\alpha > 0$ as $x \to \infty$, then for all $A > 0$,*

$$S_n(f, x) - f(x) = O(\omega_{2A}(f, n^{-1/2})), \qquad x \in [0, A], \tag{1.2}$$

where

$$\omega_A(f, \delta) \equiv \sup\{|f(x+t) - f(x)\colon |t| \leqslant \delta, x \in [0, A]\}.$$

It is easy to see, by considering the function $f(t) = |t - x|$ at the point $t = x$ $(x > 0)$, that this result is essentially the best possible.

This result was improved by Hermann [4] in 1977. He showed that (1.2) holds if $f(t) = O(t^{\alpha t})$ $(\alpha > 0)$. He proved in the same paper that the series in (1.1) does not converge if $f(t) \geqslant t^{\phi(t) \cdot t}$, where $\phi(t)$ is any monotonically increasing function such that $\lim_{t \to \infty} \phi(t) = \infty$.

In this paper, we shall study $S_n(f, x)$ for functions of bounded variation on every finite subinterval of $[0, \infty)$ and prove that $S_n(f, x)$ converges to $\frac{1}{2}(f(x+0) + f(x-0))$ under Hermann's condition on the magnitude of f by giving quantitative estimates of the rate of convergence. We shall also prove that our estimates are essentially best possible.

2. RESULTS AND REMARKS

Our main result may be stated as follows

THEOREM. *Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$ and let $f(t) = O(t^{\alpha t})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0, \infty)$ is irrational, then for n sufficiently large we have*

$$\begin{aligned} |S_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-))| \leqslant{} & \frac{(3+x)x^{-1}}{n} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + x/\sqrt{k}}(g_x) \\ & + \frac{O(x^{-1/2})}{n^{1/2}} |f(x+) - f(x-)| \\ & + O(1)(4x)^{4\alpha x}(nx)^{-1/2} \left(\frac{e}{4}\right)^{nx}, \end{aligned} \tag{2.1}$$

where $V_a^b(g)$ is total variation of g on $[a, b]$ and

$$\begin{aligned} g_x(t) &= f(t) - f(x+), && x < t < \infty, \\ &= 0, && t = x, \\ &= f(t) - f(x-), && 0 \leqslant t < x. \end{aligned}$$

The right-hand side of (2.1) converges to zero as $n \to \infty$ since the continuity of $g_x(t)$ at $t = x$ implies that

$$V_{x-\delta}^{x+\delta}(g_x) \to 0 \qquad (\delta \to 0+).$$

Remarks. If f is not constant in any neighborhood of x then

$$(4x)^{4\alpha x}(nx)^{-1/2}\left(\frac{e}{4}\right)^{nx} \leqslant \frac{1}{nx} V_0^{2x}(g_x) \tag{2.2}$$

for n sufficiently large. So in that case (2.1) becomes

$$|S_n(f,x) - \frac{1}{2}(f(x+) + f(x-)| \leqslant \frac{(4+x)\,x^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{O(x^{-1/2})}{n^{1/2}} |f(x+) - f(x-)|. \tag{2.3}$$

If, in addition, f is continuous at x then (2.3) can be further simplified to

$$|S_n(f,x) - f(x)| \leqslant \frac{(4+x)\,x^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f) \tag{2.4}$$

for sufficiently large n.

If, however, f is constant in some neighborhood of x, then since $V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f) = 0$ for all except a finite number of k's, we have

$$\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f) = c$$

for some positive constant c (depending on x). Using this and (2.2), (2.1) becomes

$$|S_n(f,x) - f(x)| \leqslant \frac{c(4+x)}{nx} \tag{2.5}$$

for n sufficiently large.

As far as the precision of the above estimates is concerned, we can prove that (2.4) cannot be asymptotically improved. Consider the function $f(t) = |t - x|$ $(x > 0)$ at $t = x$. From (2.4) we have

$$|S_n(f,x) - f(x)| \leqslant \frac{4+x}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f).$$

Since $V_{x-\delta}^{x+\delta}(f)=2\delta$, it follows that

$$|S_n(f,x)-f(x)|\leqslant\frac{2(4+x)}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{k}}\leqslant\frac{4(4+x)}{\sqrt{n}}.$$

On the other hand, by a result of Szász [1, p. 240],

$$\begin{aligned}|S_n(f,x)-f(x)|&=\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|p_k(nx)\\&=2xe^{-nx}\frac{(nx)^{[nx]}}{[nx]!}\\&\geqslant\frac{(2x/\pi e^4)^{1/2}}{n^{1/2}}.\end{aligned}$$

Hence, for the function $f(t)=|t-x|$ $(x>0)$, we have

$$\frac{(2x/\pi e^4)^{1/2}}{n^{1/2}}\leqslant|S_n(f,x)-f(x)|\leqslant\frac{4(4+x)}{n^{1/2}}.$$

Therefore (2.4) cannot be asymptotically improved.

3. Lemmas

The proof of our theorems is based on a number of lemmas. The first lemma originated in one of the questions posed by Ramanujan in a letter of January 16, 1913, to G. H. Hardy. A complete proof of this lemma was given by Watson [5], Szegö [6], and Karamata [7].

Lemma 1. *If x is a positive integer, then*

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^x}{x!}\theta(x)=\frac{1}{2}e^x,$$

where $\theta(x)$ lies between $\frac{1}{2}$ and $\frac{1}{3}$.

For an arbitrary positive number x, we have

Lemma 2. *If x is a positive number, then*

$$e^{-x}\sum_{k=0}^{[x]}\frac{x^k}{k!}=\frac{1}{2}+O(1/\sqrt{x}).$$

Proof of Lemma 2. Set $n = [x]$. Define a function $\psi(t)$ on $[n, n+1)$ as

$$\psi(t) = e^{-t} \sum_{k=0}^{n} \frac{t^k}{k!}, \qquad t \in [n, n+1).$$

Since

$$\psi'(t) = -e^{-t} \frac{t^n}{n!} < 0,$$

if $n \leqslant t < n+1$ we have

$$\psi((n+1)-) \leqslant \psi(t) \leqslant \psi(n).$$

In particular

$$\psi((n+1)-) \leqslant e^{-x} \sum_{k=0}^{[x]} \frac{x^k}{k!} \leqslant \psi(n).$$

By Lemma 1,

$$\begin{aligned}\psi(n) &= e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = e^{-[x]} \sum_{k=0}^{[x]} \frac{[x]^k}{k!} \\ &= \frac{1}{2} + e^{-[x]} \frac{[x]^{[x]}}{[x]!} (1 - \theta([x])), \\ \psi((n+1)-) &= e^{-(n+1)} \sum_{k=0}^{n} \frac{(n+1)^k}{k!} = e^{-([x]+1)} \sum_{k=0}^{[x]} \frac{([x]+1)^k}{k!} \\ &= \frac{1}{2} - e^{-([x]+1)} \frac{([x]+1)^{[x]+1}}{([x]+1)!} \theta([x]+1),\end{aligned}$$

also

$$e^{-k} \frac{k^k}{k!} \leqslant \frac{1}{\sqrt{2\pi k}};$$

hence

$$\frac{1}{2} - O\left(\frac{1}{\sqrt{x}}\right) \theta([x]+1) \leqslant e^{-x} \sum_{k=0}^{[x]} \frac{x^k}{k!} \leqslant \frac{1}{2} + O\left(\frac{1}{\sqrt{x}}\right) (1 - \theta([x])).$$

Lemma 2 now follows immediately from the fact that $\theta(x)$ is bounded.

Lemma 2 has an evident application in probability. The function $S_n(f, x)$

corresponds to the Poisson distribution, using the terminology of probability theory; the distribution function is

$$P(X < x) = \sum_{k=0}^{[x]} \frac{e^{-\alpha}\alpha^k}{k!}$$

with parameter $\alpha > 0$. If $\alpha = nx$ for some $x > 0$, then, by Lemma 2,

$$P(X < nx) = \sum_{k=0}^{[nx]} \frac{e^{nx}(nx)^k}{k!} = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right).$$

This result cannot be proved directly by applying the Central Limit Theorem; there is a similar result when x is a positive integer (see, e.g., [8, p. 302]).

The following lemma was proved by Szász (see [1, p. 239]).

LEMMA 3. *If x is a positive number, then*

$$e^{-x} \sum_{|k-x| \geqslant \delta} \frac{x^k}{k!} \leqslant \frac{x}{\delta^2}.$$

Lemma 4 is a Ramanujan-type result. The second part will not be needed in the proof of our theorems. It is given here merely because of its own interest.

LEMMA 4. (i) *If $2x$ is a positive integer then*

$$\sum_{k=2x+1}^{\infty} \frac{x^k}{k!} = \delta(x) \frac{x^{2x}}{(2x)!},$$

where $\delta(x)$ lies between $2\sqrt{e} - 3$ and 1.

(ii) *If x is a positive integer, then*

$$\alpha(x) \frac{x^x}{x!} + \frac{x^{x+1}}{(x+1)!} + \cdots + \frac{x^{2x-1}}{(2x-1)!} + \frac{x^{2x}}{(2x)!} \beta(x) = \frac{1}{2} e^x,$$

where $\alpha(x)$ lies between $\frac{1}{2}$ and $\frac{2}{3}$ and $\beta(x)$ lies between $2(\sqrt{e} - 1)$ and 2.

Proof of Lemma 4. It is easy to see that, when $2x$ is a positive integer,

$$\delta(x) = \frac{e^x - \sum_{k=0}^{2x} x^k/k!}{x^{2x}/(2x)!} = \frac{x}{2x+1} + \frac{x^2}{(2x+2)(2x+1)} + \frac{x^3}{(2x+3)(2x+2)(2x+1)} + \cdots.$$

We shall adopt this equality as the definition of $\delta(x)$ for all $x \geqslant \frac{1}{2}$. It is obvious that $\delta(\frac{1}{2}) = 2\sqrt{e} - 3$ and $\lim_{x\to\infty} \delta(x) = 1$. Therefore (i) will be proved if we can show that $\delta(x)$ is an increasing function. However, this follows immediately from the fact that

$$\frac{x^j}{\prod_{i=1}^{j}(2x+i)} \leqslant \frac{y^j}{\prod_{i=1}^{j}(2y+i)}, \qquad j = 1, 2, 3, \ldots,$$

if $x \leqslant y$.

Part (ii) is a direct consequence of Lemma 1 and (i).

LEMMA 5. *If* $x \geqslant \frac{1}{2}$, *then*

$$c_1 \frac{1}{\sqrt{x}} \left(\frac{e}{4}\right)^x \leqslant e^{-x} \sum_{k>2x} \frac{x^k}{k!} \leqslant c_2 \frac{1}{\sqrt{x}} \left(\frac{e}{4}\right)^x,$$

where $c_1 = (2\sqrt{e} - 3)/2\sqrt{\pi e}$ *and* $c_2 = \sqrt{e/4\pi}$.

Proof of Lemma 5. First, let us assume that $n \leqslant x < n + \frac{1}{2}$, where n is a positive integer. Define a function $\psi(t)$ on $[n, n+\frac{1}{2})$ as

$$\psi(t) = e^{-t} \sum_{k=2n+1}^{\infty} \frac{t^k}{k!}, \qquad t \in [n, n+\tfrac{1}{2}).$$

Since

$$\psi'(t) = e^{-t} \frac{t^{2n}}{(2n)!} > 0,$$

if $n \leqslant t < n + \frac{1}{2}$, we have

$$\psi(n) \leqslant \psi(t) \leqslant \psi((n+\tfrac{1}{2})-).$$

In particular

$$\psi(n) \leqslant e^{-x} \sum_{k>2x} \frac{x^k}{k!} \leqslant \psi\left(\left(n+\frac{1}{2}\right)-\right).$$

By Lemma 4 and Stirling's formula,

$$\psi(n) = e^{-n} \sum_{k=2n+1}^{\infty} \frac{n^k}{k!} = e^{-[x]} \sum_{k=2[x]+1}^{\infty} \frac{[x]^k}{k!}$$
$$= e^{-[x]}\, \delta([x]) \frac{[x]^{2[x]}}{(2[x])!}$$

$$\geqslant \frac{1}{\sqrt{[x]}}\left(\frac{e}{4}\right)^{[x]} \frac{1}{4\sqrt{\pi}}\,\delta([x])$$

$$\geqslant \frac{1}{\sqrt{x}}\left(\frac{e}{4}\right)^{x} \frac{1}{2\sqrt{\pi e}}\,\delta([x]);$$

$$\psi\left(\left(n+\frac{1}{2}\right)-\right) = e^{-n-1/2} \sum_{k=2n+1}^{\infty} \frac{(n+1/2)^k}{k!}$$

$$= e^{-[x]-1/2} \sum_{k=2[x]+1}^{\infty} \frac{([x]+1/2)^k}{k}$$

$$= e^{-[x]-1/2}\left(1+\delta\left([x]+\frac{1}{2}\right)\right)\frac{([x]+1/2)^{2[x]+1}}{(2[x]+1)!}$$

$$\leqslant \frac{1}{\sqrt{[x]+1/2}}\left(\frac{e}{4}\right)^{[x]+1/2} \frac{1}{2\sqrt{\pi}}\left(1+\delta([x]+\frac{1}{2})\right)$$

$$\leqslant \frac{1}{\sqrt{x}}\left(\frac{e}{4}\right)^{x} \frac{\sqrt{e}}{4\sqrt{\pi}}\left(1+\delta\left([x]+\frac{1}{2}\right)\right).$$

Therefore, when $n \leqslant x < n+\frac{1}{2}$, as $\delta(x)$ lies between $2\sqrt{e}-3$ and 1, we have

$$\frac{2\sqrt{e}-3}{2\sqrt{\pi e}}\frac{1}{\sqrt{x}}\left(\frac{e}{4}\right)^{x} \leqslant e^{-x}\sum_{k>2x}\frac{x^k}{k!} \leqslant \sqrt{\frac{e}{4\pi}}\cdot\frac{1}{\sqrt{x}}\left(\frac{e}{4}\right)^{x}. \tag{3.1}$$

Next, if $n+\frac{1}{2} \leqslant x < n+1$ for some nonnegative integer n, we define $\psi(t)$ on $[n+\frac{1}{2}, n+1)$ as

$$\psi(t) = e^{-t}\sum_{k=2n+2}^{\infty}\frac{t^k}{k!}, \qquad t \in [n+\tfrac{1}{2}, n+1).$$

Along the same lines, we can prove that, when $n+\frac{1}{2} \leqslant x < n+1$ for some nonnegative integer n, $e^{-x}\sum_{k>2x} x^k/k!$ satisfies again (3.1). This completes the proof.

The last lemma of this section is similar to a result in Hermann's paper [4], but has a more precise estimate.

LEMMA 6. *For any fixed positive numbers α and x,*

$$\sum_{k>2x}\left(\frac{k}{n}\right)^{\alpha(k/n)} p_k(x) \leqslant \frac{3}{2}\left(\frac{2x+1}{n}\right)^{\alpha(2x+1)/n}\frac{1}{\sqrt{\pi x}}\left(\frac{e}{4}\right)^{x}$$

if n is sufficiently large.

Proof of Lemma 6. Let

$$b_k = \left(\frac{k}{n}\right)^{\alpha(k/n)} p_k(x), \qquad k > 2x.$$

By a simple calculation we can show that

$$\frac{b_{k+1}}{b_k} \leqslant \frac{2}{3}$$

if n is sufficiently large. Hence, if $k > 2x$,

$$\sum_{k>2x} \left(\frac{k}{n}\right)^{\alpha(k/n)} p_k(x) \leqslant 3b_{[2x]+1} = 3\left(\frac{[2x]+1}{n}\right)^{\alpha([2x]+1)/n} p_{[2x]+1}(x)$$

$$\leqslant 3\left(\frac{2x+1}{n}\right)^{\alpha(2x+1)/n} p_{[2x]+1}(x). \tag{3.2}$$

By Stirling's formula

$$p_{[2x]+1}(x) \leqslant \frac{1}{\sqrt{4\pi x}} (e^{1-(x/([2x]+1))+\log(x/([2x]+1))})^{[2x]+1}.$$

Since the function $g(y) = 1 - y + \log y$ is increasing on $(0, \frac{1}{2}]$ and $g(\frac{1}{2}) = \frac{1}{2} - \log 2 < 0$, we have

$$p_{[2x]+1}(x) \leqslant \frac{1}{\sqrt{4\pi x}} (e^{1/2-\log 2})^{[2x]+1}$$

$$\leqslant \frac{1}{\sqrt{4\pi x}} (e^{1/2-\log 2})^{2x}$$

$$= \frac{1}{\sqrt{4\pi x}} \left(\frac{e}{4}\right)^{x}. \tag{3.3}$$

Lemma 6 follows immediately from (3.2) and (3.3).

If, in Lemma 6, we replace x by nx, then, when n is sufficiently large, we get the inequality

$$\sum_{k>2nx} \left(\frac{k}{n}\right)^{\alpha(k/n)} p_k(nx) \leqslant \frac{3}{2} (4x)^{4\alpha x} \frac{1}{\sqrt{nx\pi}} \left(\frac{e}{4}\right)^{nx}$$

which is what we really need in the proof of our theorem.

4. Proof of the Theorem

For any fixed $x \in (0, \infty)$, define g_x as

$$\begin{aligned} g_x(t) &= f(t) - f(x+), && x < t < \infty, \\ &= 0, && t = x, \\ &= f(t) - f(x-), && 0 \leqslant t < x. \end{aligned} \tag{4.1}$$

g_x is continuous at $t = x$ and inherits all the properties of f. Using (4.1), (1.1) can be written as

$$S_n(f, x) = S_n(g_x, x) + \frac{f(x+) + f(x-)}{2} + \frac{f(x+) - f(x-)}{2} \times (A_n(x) - B_n(x)),$$

where

$$A_n(x) = \sum_{k \geqslant nx} p_k(nx) = e^{-nx} \sum_{k \geqslant nx} \frac{(nx)^k}{k!},$$

$$B_n(x) = \sum_{k \leqslant nx} p_k(nx) = e^{-nx} \sum_{k \leqslant nx} \frac{(nx)^k}{k!}.$$

Hence

$$|S_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-))| \leqslant |S_n(g_x, x)| + \tfrac{1}{2} |f(x+) - f(x-)| \times |A_n(x) - B_n(x)|. \tag{4.2}$$

By Lemma 2,

$$B_n(x) = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right)$$

and

$$A_n(x) = 1 - B_n(x) = \frac{1}{2} + O\left(\frac{1}{\sqrt{nx}}\right).$$

Therefore, for the second summand on the right-hand side of (4.2) we have

$$\frac{1}{2} |f(x+) - f(x-)| \, |A_n(x) - B_n(x)| = O\left(\frac{1}{\sqrt{nx}}\right) |f(x+) - f(x-)| \tag{4.3}$$

and our theorem will be proved if we establish that

$$|S_n(g_x, x)| \leqslant \frac{(3+x)\,x^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$

$$+ O(1)(4x)^{4\alpha x}(nx)^{-1/2}\left(\frac{e}{4}\right)^{nx} \tag{4.4}$$

for suffciently large n.

To do this we first observe that $S_n(g_x, x)$ can be written as a Lebesgue–Stieltjes integral

$$S_n(g_x, x) = \int_0^\infty g_x(t)\, d_t K_n(x, t), \tag{4.5}$$

where the kernel $K_n(x, t)$ is defined by

$$\begin{aligned} K_n(x, t) &= \sum_{k \leqslant nt} p_k(nx), \qquad && 0 < t < \infty, \\ &= 0, && t = 0, \end{aligned}$$

the so-called Poisson distribution of probability. We decompose the integral on the right-hand side of (4.5) into three parts, as

$$\int_0^\infty g_x(t)\, d_t K_n(x, t) = L_n(f, x) + M_n(f, x) + R_n(f, x), \tag{4.6}$$

where

$$L_n(f, x) = \int_0^{x - x/\sqrt{n}} g_x(t)\, d_t K_n(x, t),$$

$$M_n(f, x) = \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)\, d_t K_n(x, t),$$

$$R_n(f, x) = \int_{x+x/\sqrt{n}}^{\infty} g_x(t)\, d_t K_n(x, t).$$

We shall evaluate consecutively $M_n(f, x)$, $L_n(f, x)$, and $R_n(f, x)$. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$,

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant V_{x-x/\sqrt{n}}^{x-x/\sqrt{n}}(g_x).$$

Hence

$$|M_n(f, x)| \leqslant V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \cdot \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t K_n(x, t).$$

Since

$$\int_a^b d_t K_n(x, t) \leqslant 1 \qquad \text{for any } [a, b] \subseteq [0, \infty),$$

therefore

$$|M_n(f, x)| \leqslant V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x). \tag{4.7}$$

Next, we evaluate $L_n(f, x)$. The method used here is similar to the approach used by Bojanic and Vuilleumier [9].

Using partial integration with $y = x - x/\sqrt{n}$, we have

$$L_n(f, x) = g_x(y+) K_n(x, y+) - \int_0^y \hat{K}_n(x, t)\, d_t\, g_x(t),$$

where $\hat{K}_n(x, t)$ is the normalized form of $K_n(x, t)$. If $0 < y < \infty$, then $K_n(x, y+) = K_n(x, y)$ and

$$|\, g_x(y+)| = |\, g_x(y+) - g_x(x)| \leqslant V_{y+}^x(g_x),$$

where $V_{y+}^x(g_x) = \lim_{\varepsilon \to 0+} V_{y+\varepsilon}^x(g_x)$. Therefore

$$|L_n(f, x)| \leqslant V_{y+}^x(g_x) K_n(x, y) + \int_0^y \hat{K}_n(x, t)\, d_t(-V_t^x(g_x)).$$

Since $\hat{K}_n(x, t) \leqslant K_n(x, t)$ on $(0, \infty)$ and since by Lemma 3,

$$K_n(x, t) = \sum_{k \leqslant nt} p_k(nx) \leqslant x/n(t-x)^2, \qquad 0 \leqslant t < x,$$

we have

$$|L_n(f, x)| \leqslant V_{y+}^x(g_x) K_n(x, y) + \frac{x}{n} \int_{0+}^y \frac{1}{(t-x)^2}\, dt(-V_t^x(g_x))$$
$$+ \frac{1}{2} \cdot e^{-nx} V_0^{0+}(g_x).$$

Since, for $nx > 0$,

$$e^{-nx} < \frac{1}{nx}$$

and

$$\frac{x}{n}\int_{0+}^{y}\frac{1}{(t-x)^2}\,d_t(-V_t^x(g_x))+\frac{1}{nx}\,V_0^{0+}(g_x)$$
$$=\frac{x}{n}\int_{0}^{y}\frac{1}{(t-x)^2}\,d_t(-V_t^x(g_x)),$$

it follows that

$$|L_n(f_r, x)|\leqslant V_{y+}^x(g_x)\,\frac{x}{n(x-y)^2}+\frac{x}{n}\int_0^y\frac{1}{(x-t)^2}\,d_t(-V_t^x(g_x)).$$

Using partial integration again, we have

$$\int_0^y\frac{1}{(x-t)^2}\,d_t(-V_t^x(g_x))=-\frac{V_{y+}^x(g_x)}{(x-y)^2}+\frac{V_0^x(g_x)}{x^2}$$
$$+2\int_0^y V_t^x(g_x)\,\frac{dt}{(x-t)^3}.$$

Hence

$$|L_n(f, x)|\leqslant\frac{x}{n}\left(\frac{V_0^x(g_x)}{x^2}+2\int_0^y V_t^x(g_x)\,\frac{dt}{(x-t)^3}\right).$$

Replacing the variable t in the last integral by $x-x/\sqrt{t}$, we find that

$$\int_0^{x-x/\sqrt{n}} V_t^x(g_x)\,\frac{dt}{(x-t)^3}=\frac{1}{2x^2}\int_1^n V_{x-x/\sqrt{t}}^x(g_x)\,dt$$
$$\leqslant\frac{1}{2x^2}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x).$$

Thus

$$|L_n(f, x)|\leqslant\frac{1}{nx}\left(V_0^x(g_x)+\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x)\right)$$
$$\leqslant\frac{2}{nx}\sum_{k=1}^{n} V_{x-x/\sqrt{k}}^x(g_x). \tag{4.8}$$

Finally, we evaluate $R_n(f, x)$. Let $z=x+x/\sqrt{n}$ and define $Q_n(x, t)$ on $[0, 2x]$ as

$$Q_n(x, t)=1-P_n(x, t-),\qquad 0\leqslant t<2x,$$
$$=0,\qquad t=2x.$$

Then

$$R_n(f, x) = -\int_z^{2x} g_x(t)\, d_t Q_n(x, t) - g_x(2x) \sum_{k > 2nx} p_k(nx)$$
$$+ \int_{2x}^{\infty} g_x(t)\, d_t K_n(x, t)$$
$$= R_{1n} + R_{2n} + R_{3n}. \tag{4.9}$$

Using partial integration for the first term on the right-hand side of (4.9), we get

$$R_{1n} = g_x(z-)\, Q_n(x, z-) + \int_z^{2x} \hat{Q}_n(x, t)\, d_t\, g_x(t),$$

where $\hat{Q}_n(x, t)$ is the normalized form of $Q_n(x, t)$. Since $Q_n(x, z-) = Q_n(x, z)$, $0 \leqslant z < 1$, and $|g_x(z-)| \leqslant V_x^{z-}(g_x)$, we have

$$|R_{1n}| \leqslant V_x^{z-}(g_x)\, Q_n(x, z) + \int_z^{2x} \hat{Q}_n(x, t)\, d_t V_x^t(g_x).$$

Since by Lemma 3

$$Q_n(x, t) = \sum_{k \geqslant nt} p_k(nx) \leqslant \frac{x}{n(t-x)^2}, \qquad x < t < 2x,$$

and since $\hat{Q}_n(x, t) \leqslant Q_n(x, t)$ on $[0, 2x)$, we have

$$|R_{1n}| \leqslant V_x^{z-}(g_x) \frac{x}{n(z-x)^2} + \frac{x}{n} \int_z^{2x-} \frac{1}{(t-x)^2}\, d_t V_x^t(g_x)$$
$$+ \frac{1}{2} \Big(\sum_{k > 2nx} p_k(nx)\Big)\, V_{2x-}^{2x}(g_x).$$

Next, the inequality

$$\frac{1}{2} \left(\sum_{k > 2nx} p_k(nx) \right) V_{2x-}^{2x}(g_x) \leqslant \frac{1}{\sqrt{nx\pi}} \left(\frac{e}{4} \right)^{nx} V_{2x-}^{2x}(g_x)$$
$$\leqslant \frac{1}{nx} V_{2x-}^{2x}(g_x)$$

which follows from Lemma 5 and the identity

$$\frac{x}{n} \int_z^{2x-} \frac{1}{(t-x)^2}\, d_t V_x^t(g_x) + \frac{1}{nx} V_{2x-}^{2x}(g_x) = \int_z^{2x} \frac{1}{(t-x)^2}\, d_t V_x^t(g_x)$$

imply

$$|R_{1n}| \leqslant V_x^{z-}(g_x) \frac{x}{n(z-x)^2} + \frac{x}{n} \int_z^{2x} \frac{1}{(x-t)^2} \, d_t V_x^t(g_x).$$

Integrating by parts the last integral, we get

$$\int_z^{2x} \frac{1}{(t-x)^2} \, d_t V_x^t(g_x) = \frac{V_x^{2x}(g_x)}{x^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} + 2 \int_z^{2x} V_x^t(g_x) \frac{dt}{(t-x)^3}.$$

Hence

$$|R_{1n}| \leqslant \frac{x}{n} \left(\frac{V_x^{2x}(g_x)}{x^2} + 2 \int_z^{2x} V_x^t(g_x) \frac{dt}{(t-x)^3} \right).$$

Replacing the variable in the last integral by $x + x/\sqrt{t}$, we find that

$$\int_z^{2x} V_x^t(g_x) \frac{dt}{(t-x)^3} = \frac{1}{2x^2} \int_1^n V_x^{x+x/\sqrt{t}}(g_x) \, dt \leqslant \frac{1}{2x} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

Therefore

$$|R_{1n}| \leqslant \frac{1}{nx} \left(V_x^{2x}(g_x) + \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x) \right) \leqslant \frac{2}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{4.10}$$

The evaluation of R_{2n} is relatively easy. By Lemma 5, we have

$$|R_{2n}| \leqslant |g_x(2x)| \frac{1}{\sqrt{nx\pi}\,(4/e)^{nx}}.$$

But

$$|g_x(2x)| \leqslant \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x)$$

and

$$\frac{1}{\sqrt{nx\pi}\,(4/e)^{nx}} \leqslant \frac{1}{nx}.$$

Consequently

$$|R_{2n}| \leqslant \frac{1}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{4.11}$$

Finally, by Lemma 6 and the assumption that $f(t) = O(t^{\alpha t})$ ($\alpha > 0$) as $t \to \infty$, we see that for n sufficiently large,

$$\begin{aligned} |R_{3n}| &\leqslant M \sum_{k > 2nx} \left(\frac{k}{n}\right)^{\alpha(k/n)} p_k(nx) \\ &\leqslant \frac{3M}{2} (4x)^{4\alpha x} \frac{1}{\sqrt{nx\pi}} \left(\frac{e}{4}\right)^{nx} \end{aligned} \tag{4.12}$$

for some positive constant M.

Hence, from (4.10), (4.11), and (4.12), we obtain for n sufficiently large,

$$|R_n(f, x)| \leqslant \frac{3}{nx} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x) + O(1)(4x)^{4\alpha x}(nx)^{-1/2}\left(\frac{e}{4}\right)^{nx}. \tag{4.13}$$

Equation (4.4) now folows from (4.5)–(4.8), (4.13), and the fact that

$$V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leqslant \frac{1}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x).$$

References

1. O. Szász,, Generalization of S. Bernstein's polynomials to the infinite interval, *J. Res. Nat. Bur. Standards*, Sect. B **45** (1950), 239–245.
2. G. Mirakyan, Approximation des fonctions continues au moyen de polynômes de la forme $e^{-nx} \sum_{k=0}^{m} C_{k,n} x^k$, *C. R. (Doklady) Acad. Sci. URSS* **31** (1941), 201–205.
3. J. Gróf, A Szász Ottó-féle operátor approximaciós tulajdonsagairól, Mat III, *Oszt. Közl.* **20** (1971), 35–44. [Hungarian]
4. T. Hermann, Approximation of unbounded functions on unbounded interval, *Acta Math. Acad. Sci. Hungar.* **29** (3–4) (1977), 393–398.
5. G. N. Watson, Theorems stated by Ramanujan. V. Approximations connected with e^x, *Proc. London Math. Soc.* (2) **29** (1928), 293–308.
6. G. Szegö, Ueber einige von S. Ramanujan gestellte Aufgaben, *J. London Math. Soc.* **3** (1928), 225–232.
7. J. Karamata, Sur quelques problèmes posés par Ramanujan, *J. Indian Math. Soc.* (*N.S.*) **24** (1960), 343–365.
8. B. V. Gnedenko, "The Theory of Probability," Chelsea, New York, 1962.
9. R. Bojanic and M. Vuilleumier, On the rate of convergence of Fourier–Legendre series of functions of bounded variation, *J. Approx. Theory* **31** (1981), 67–79.